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ON THE LIGHT CONE SINGULARITY OF THE THERMAL EFFECTIVE EXPANSION

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ABSTRACT

We consider a scalar massless quantum field model, at finite temperature T, both renormalizable and asymptotically free. Focussing on the singular structure of the effective perturbation theory about the light cone, several new insights are put forth, regarding the interplay between hard thermal loop resummation and the overall compensation of collinear singularities.

1. Introduction

The inherent non perturbative character of finite temperature quantum field theories has been recognized a few years ago on the basis of general group theoretical arguments [1]. At the same time, the formal perturbative series themselves were shown to necessitate a full redefinition of their original form so as to yield (hopefully) sensical results. This is the resummation program devised by Braaten and Pisarski and also by Frenkel and Taylor [2], hereafter referred to as "effective theory" for short.

In order not to be just an academic recipe, such a program implicitly requires that the original and/or re-organised perturbative series be infrared safe. In the recent years, a great deal of efforts has been devoted to the study of the infrared structure of the original perturbative series, with the conclusion that, roughly speaking, the situation at non zero temperature ($T \neq 0$) was not worse, globally, than at zero temperature ($T = 0$) [3].

But re-defining a perturbative series is rarely trivial an operation. Properties which were known to hold true for the original series may become much harder to control in the redefined one. In our opinion, the collinear singularity recently found in thermal QCD by using the effective perturbative series [4,5] might reveal a salient illustration of this fact. Note that we are aware that this point of view differs, at first sight, from the ones adopted in the solutions proposed in [6] and [7]. Very recently, though, it has been realized [18], that some unexpected connection could exist with the analysis proposed in [6]. In this paper we will restrict ourselves to a scalar model and analyse the singular structure of the effective theory in much details.

A preliminary report of the present work appeared in [8], which is here corrected, completed and enlarged. The paper is organized as follows. Section 2 is an introduction to the model, and the necessary elements of the particular real time formalism which is used, are given. In section 3, the singularity structure of a whole series of diagrams is investigated, deferring some lengthy details to an appendix. The results of this section rely on an approximation which consists in keeping only the potentially most singular part of the hard thermal loop (HTL) self energy. This approximation gets completed in section 4, where the other terms are taken into account. Further consequences are drawn in sections 5 and 6, respectively related to the problem of the $T = 0$ and $T \neq 0$ singularity mixing, and to the possibility of an unambiguous renormalisation constant definition in a thermal context. Eventually, a conclusion gathers our results.

2. The model

In order to get rid of unessential complications, we deal with the toy model provided by a massless hermitian scalar quantum field. It is endowed with a cubic self interaction, of coupling strength g, renormalisable at $D = 6$ space-time dimensions. Furthermore, its quanta are assumed to form a plasma in thermodynamical equilibrium at temperature T. Infrared as well as ultraviolet singularities are taken care of by working at $D = 6 \pm 2\varepsilon$ space-time dimensions respectively. Also, we will use the convention of upper case letters for D-momenta and lower case ones for their components, i.e, $P = (p_0, \vec{p})$.

Use will be made of the material and results of a previous work [9] where a calculation performed at second non trivial order of bare perturbation theory, $O(g^4)$, was able to display the overall compensation of infrared and collinear singularities for some particular topology.

The process under consideration consists in a "Higgs" particle of D-momentum $Q = (q, \vec{0})$ in the rest-frame of a plasma in equilibrium at temperature T. But, contrary to the case studied in [9], this D-momentum is here assumed to be soft, that is $q = O(gT)$. Also, our analysis will take place in the framework of the real time formalism, within "the basis" provided by the retarded and advanced free field functions, $\Delta_R(K)$ and $\Delta_A(K)$ respectively [5]. At any step, though, we have checked that the same expressions are obtained in the more customary real time formalism of the Kobes and Semenoff rules (the "1/2" formalism, say).

The real part of the thermal one loop self energy reads [9]

$$\mathrm{Re}\Sigma_T(P) = \frac{2g^2}{(4\pi)^{3+\varepsilon}\Gamma(2+\varepsilon)} \int_0^\infty \mathrm{d}k\; k^{D-3} n(k) \sum_{\eta=\pm 1} \mathbf{P} \int_{-1}^{1} \mathrm{d}x \frac{(1-x^2)^{1+\varepsilon}}{P^2 + 2k(\eta p_0 - px)} \tag{2.1}$$

where a combinational weight factor 1/2 is taken into account, and where **P** stands for a principal value prescription. Also, throughout the calculations, use will be made of the following relations for statistical factors

$$n(k_0) = (e^{\beta k_0} - 1)^{-1},\ n^B(k_0) = (e^{\beta |k_0|} - 1)^{-1},\ \varepsilon(k_0)(1 + 2n(k_0)) = 1 + 2n^B(k_0) \tag{2.2}$$

The phase-space domain where collinear singularities come from has been recognized to be determined by the condition $P^2 << p^2 << T^2$. In this domain, the function $\{Re\Sigma_T\}$, that we choose to describe in terms of the two independent variables P^2 and p^2, is approximated by a real function A, defined for all P^2 values

$$A(P^2, p^2) = -m^2\ \frac{1}{\varepsilon}\ \frac{P^2}{p^2}\left\{1 - \frac{1 + 2^{1+\varepsilon}\varepsilon}{4^\varepsilon}\ |\frac{P^2}{p^2}|^{\ \varepsilon}\ \left(\Theta(P^2) + \cos(\pi\varepsilon)\Theta(-P^2)\right)\right\} \tag{2.3}$$

where m^2, the thermal mass squared, is given by

$$m^2 = 2\frac{g^2}{(4\pi)^3}\zeta(2 + 2\varepsilon)T^{2+2\varepsilon} \tag{2.4}$$

Expression (2.3) displays the full HTL (that is, leading, order g^2T^2 part) entailed in the real part of the thermal self energy function. Only for space-like values of its argument P^2, does the thermal self energy function develop an imaginary part of same leading order g^2T^2. This corresponds to the so called "Landau damping mechanism", and one can write eventually

$$\Sigma_F = A + iB\ , \qquad B(P^2, p^2) = -\pi m^2(-\frac{P^2}{p^2})^{1+\varepsilon}\Theta(-P^2)\frac{1 + 2^{1+\varepsilon}\varepsilon}{4^\varepsilon}\ \frac{\sin(\pi\varepsilon)}{\pi\varepsilon} \tag{2.5}$$

where Σ_F denotes the Feynman self energy. Indeed, the functions A and B can be seen as obtained through the usual Feynman prescription $\{p_0 \longrightarrow p_0 + i\eta p_0\}$, out of *one* analytical self energy function Σ defined in the complex energy plane

$$A = \lim_{\eta=0}\ \frac{1}{2}\left(\Sigma(p_0 + i\eta p_0, p) + \Sigma(p_0 - i\eta p_0, p)\right)$$

$$B = \lim_{\eta=0}\ \frac{1}{2i}\left(\Sigma(p_0 + i\eta p_0, p) - \Sigma(p_0 - i\eta p_0, p)\right)$$

with Σ the light cone approximate self energy

$$\Sigma = -m^2\ \frac{1}{\varepsilon}\ \frac{P^2}{p^2}\left\{1 - \frac{1 + 2^{1+\varepsilon}\varepsilon}{4^\varepsilon}\left(\frac{P^2}{p^2}\right)^{\varepsilon}\right\} \tag{2.6}$$

Note that in (2.6), the limit $\varepsilon = 0$ can be safely taken and that the thermal self energy function is a regular function of P^2 in any neighbourhood of the light cone. One gets effectively

$$\Sigma(P^2, p^2) = -2m^2\frac{P^2}{p^2}\left(\frac{1}{2}\ln(\frac{4p^2}{P^2}) - 1\right) \tag{2.7}$$

which, expressed in terms of the variables P^2 and p^2, is nothing but the more familiar form [12] of the HTL self energy of the model, with its second kind Legendre function $Q_1(p_0, p)$ approximate form at $|p_0| \sim p$. In the R/A real time formalism we will be using, retarded and advanced functions must be used. They are simply related to the Feynman ones by the set of relations

$$\Sigma_R(P) = \Theta(p_0)\Sigma_F(P) + \Theta(-p_0)\Sigma_F^*(P), \quad \Sigma_A(P) = \Theta(p_0)\Sigma_F^*(P) + \Theta(-p_0)\Sigma_F(P) \tag{2.8}$$

In the region $\mathcal{D}$ where one has $P^2 << p^2 << T^2$ and $O(p_\mu) = gT$, effective propagators ${}^*\Delta_\alpha(P)$ must be used so as to get the full leading order g^2T^2 correction to the free quantities, with

$${}^*\Delta_\alpha(P) = \frac{i}{P^2 - \Sigma_\alpha(P^2, p^2) + i\varepsilon_\alpha p_0}\ , \qquad \alpha = R, A. \tag{2.9}$$

A point to be stressed is that the domain which requires that effective perturbation theory be used, is also the domain where mass (or collinear) singularities stem from, thereby opening up the very "window" where the light cone singular behaviour of the effective thermal expansion must be studied.

An important remark is in order. Admittedly, there is no HTL in the three-point function of the $(g\varphi^3)_6$ theory [10], in the sense that g^2T^2 leading parts, eventually, cancel out, leaving order $g^2T^{3/2}$ pieces at most. However, in [10], not all the terms were considered and, more generally, it should be noted that this statement is not sufficient for ignoring the one loop three point function. For example, it has been stressed recently [7] that, over collinear configuration of external soft momenta, a special enhancement of bare orders of magnitude, results out of collinear singularities, leading to a breakdown of the HTL resummation program. However, besides the new internal difficulties encountered by the improved HTL resummation program itself [24], it is important to realize that such a situation does not show up here. This is basically because the Higgs particle has timelike D-momentum $Q=(q,\vec{0})$. Explicit calculation then shows that no collinear singularity ever develop in that case. Now, a wider analysis of the three point function is certainly worth undertaking, but falls beyond the scope of the present article where one loop vertex corrections will not be considered.

3- Singular structure

We begin with considering the case of N one-loop self energy insertions along the P-line as depicted on Fig.1, with $N'=0$. Then, the retarded "polarisation tensor" $\Pi^{(N)}_{RR}(Q)$ of the"Higgs particle" (hereafter written $\Pi^{(N)}_{R}(Q)$ for short) can be shown to admit the following expression (see (61) in [5])

$$\Pi^{(N)}_R(Q)=-i\int\frac{\mathrm{d}^DP}{(2\pi)^D}(1+2n(p_0))\ \mathrm{Disc}_P\left(\Delta^{(N)}_R(P)\Delta_R(P')V^2_{RRA}(P,Q,-P')\right)\tag{3.1}$$

where V_{RRA} is the bare vertex with two external lines of the retarded type, corresponding to D-momenta P and Q, and one advanced external line of D-momentum $-P'$. The convention for the flow of momenta is that their sum vanishes: $P+Q+(-P')=0$. At zeroth order, V_{RRA} is simply given by the "electric charge", e, which couples the Higgs particle to the hermitian scalar field. Eventually, the prescription "$\mathrm{Disc_P}$" means that the discontinuity in the variable p_0 is to be taken. In the $\{R/A\}$ real time formalism we are using, this is simply achieved by writing

$$\mathrm{Disc_P}F_{R\beta\delta}(P,Q,R)=F_{R\beta\delta}(P,Q,R)-F_{A\beta\delta}(P,Q,R)\tag{3.2}$$

with β and δ any of the two R,A possibilities.

In above equation (3.1), $\Delta^{(N)}_R$ denotes a N one-loop self energy corrected free field function, and $\Pi^{(N)}_R(Q)$, the ensuing corrected tensor. Omitting unessential factors, its imaginary part reads

$$\mathrm{Im}\Pi^{(N)}_R(Q)=\int\mathrm{d}^DP(1+2n(p_0))\varepsilon(p'_0)\delta(P'^2)\{\frac{(-1)^N}{N!}\pi\varepsilon(p_0)\delta^{(N)}(P^2)\mathrm{Re}(\Sigma^N_R(P))-\mathbf{P}\frac{\mathrm{Im}(\Sigma^N_R(P))}{(P^2)^{N+1}}\}\tag{3.3}$$

where $\varepsilon(p_0)$ is the sign of p_0, and $\delta^{(N)}$ the N^{th} derivative of the δ-distribution. In view of equations (2.3), (2.5) and of their bounded P-domain, note that the real and imaginary parts of the $\{\Sigma^{HTL}_R(P)\}^N$-insertions are acceptable test-functions, yielding well defined expressions when acted upon with $\delta^{(N)}(P^2)$ and $\mathbf{P}/(P^2)^{N+1}$ distributions respectively. How things are working out is instructive to observe. At $N=1$, we have from the $\delta^{(1)}(P^2)$-term

$$\frac{-e^2}{(2\pi)^{D-1}}\int\mathrm{d}^{D-1}p\int\frac{\mathrm{d}P^2}{2p_0}(1+2n(p_0))\varepsilon(p_0)$$
$$\times\ \varepsilon(p'_0)\delta(P'^2)\pi\delta(P^2)\frac{\mathrm{d}}{\mathrm{d}P^2}A(P^2,p^2)\tag{3.4}$$

where $|p_0(P^2,p^2)| = \sqrt{P^2+p^2}$. Substituing (2.3), we get immediately

$$-\frac{\pi e^2 m^2}{(2\pi)^{D-1}}\frac{\Omega_D}{4q}(\frac{q}{2})(1+2n^B(\frac{q}{2}))(\frac{1}{\varepsilon}) \tag{3.5}$$

where Ω_D is the total solid angle of the model. The singular piece coming from the term involving the principal value, more involved, is

$$-\frac{e^2}{(2\pi)^{D-1}}\frac{1+2^{1+\varepsilon}\varepsilon}{4^\varepsilon}\ \frac{\sin(\pi\varepsilon)}{\pi\varepsilon}\int\frac{\mathrm{d}^{D-1}p}{2p}(1+2n(p))\varepsilon(p'_0)\delta(P'^2)$$
$$\times\ \mathbf{P}\int\frac{\mathrm{d}P^2}{(P^2)^2}\left(-\pi m^2(-\frac{P^2}{p^2})^{1+\varepsilon}\right)\Theta(-P^2) \tag{3.6}$$

where (2.5) has been used. By integrating over P^2 first, which yields a ε^{-1} mass singularity, and then over p, one gets

$$+\frac{\pi e^2 m^2}{(2\pi)^{D-1}}\frac{\Omega_D}{4q}(\frac{q}{2})(1+2n^B(\frac{q}{2}))(\frac{1}{\varepsilon})\ \frac{1+2^{1+\varepsilon}\varepsilon}{4^\varepsilon}\ \frac{\sin(\pi\varepsilon)}{\pi\varepsilon} \tag{3.7}$$

The sum of the two terms displays the compensation of the collinear singularity, as noticed long ago by many authors for scalar as well as for gauge theories.

At $N=2$, one has $\mathrm{Re}\Sigma_R^2(P) = A^2(P) - B^2(P)$. The function B is of the order of $(-P^2)^{1+\varepsilon}$ with ε positive, and therefore does not contribute when acted upon with the $\delta^{(2)}(P^2)$ distribution. Indeed, for exactly the same reason, only the $\{A^N\}$-part of $\mathrm{Re}\Sigma_R^N(P)$ will ever contribute at general N, because of the $\delta^{(N)}(P^2)$ distribution. In the present case, $N=2$, one gets

$$-\pi\frac{e^2}{(2\pi)^{D-1}}\frac{\Omega_D}{4q}\frac{2}{q}\left(1+2n^B(\frac{q}{2})\right)(\frac{m^2}{\varepsilon})^2 \tag{3.8}$$

which, by recalling that q is of the order of gT, is consistently seen to be on the same order as (3.5) and (3.7) at $N=1$. Now, the part involving the principal value distribution reads

$$-\pi\frac{e^2}{(2\pi)^{D-1}}\frac{1+2^{1+\varepsilon}\varepsilon}{4^\varepsilon}\frac{\sin(\pi\varepsilon)}{\pi\varepsilon}\int\mathrm{d}^{D-1}p\int\frac{\mathrm{d}P^2}{2p_0}(1+2n(p_0))\varepsilon(p'_0)\delta(P'^2)\left(-\frac{\mathbf{P}}{(P^2)^3}\right)$$
$$\times\ 2\varepsilon(p_0)\left(-\frac{m^2P^2}{\varepsilon p^2}\ (1-\frac{1+2^{1+\varepsilon}\varepsilon}{4^\varepsilon}\ \frac{\sin(\pi\varepsilon)}{\pi\varepsilon}|\frac{P^2}{p^2}|^\varepsilon\cos(\pi\varepsilon))\right)\left(-\pi m^2(-\frac{P^2}{p^2})^{1+\varepsilon}\right)\Theta(-P^2) \tag{3.9}$$

Integrating out over P^2 and p, one gets

$$+\pi\frac{e^2}{(2\pi)^{D-1}}\frac{\Omega_D}{4q}\frac{2}{q}\left(1+2n^B(\frac{q}{2})\right)\frac{1+2^{1+\varepsilon}\varepsilon}{4^\varepsilon}\ \frac{\sin(\pi\varepsilon)}{\pi\varepsilon}(-\frac{m^4}{\varepsilon})\left(\frac{1}{\varepsilon}-\frac{1+2^{1+\varepsilon}\varepsilon}{4^\varepsilon 2\varepsilon}\cos(\pi\varepsilon)\right) \tag{3.10}$$

Expanding in powers of ε the terms

$$\frac{1+2^{1+\varepsilon}\varepsilon}{4^\varepsilon}\ \frac{\sin(\pi\varepsilon)}{\pi\varepsilon}\left(\frac{1}{\varepsilon}-\frac{1+2^{1+\varepsilon}\varepsilon}{4^\varepsilon 2\varepsilon}\cos(\pi\varepsilon)\right) = \frac{1}{2\varepsilon}\ \left(1+O(\varepsilon^2)\right) \tag{3.11}$$

the sum of the two terms (3.9) and (3.10) is readily seen to display the expected singularity compensation. Indeed, inspection of the first values of N ($1 \le N \le 5$), shows that one may drop the terms $2^\varepsilon, 4^{-\varepsilon}$ appearing in (2.3) and (2.5) without spoiling at all the singularity cancellation mechanism. In what follows, we will take advantage of that simplification in order to avoid overcharging with details.

At $N=3$, one has

$$\Sigma_R^3(P^2,p^2) = (A^3-3AB^2)(P^2,p^2)\ +\ i\varepsilon(p_0)(3A^2B-B^3)(P^2,p^2) \tag{3.12}$$

Of the real part, only the $\{A^3\}$ term will contribute according to the remark above. The resulting singularity, the stronger, of order ε^{-3} is given by

$$\left((-1)^3/3!\right)\pi\varepsilon(p_0)\delta^{(3)}(P^2)\ A^3(P^2,p^2) = \pi\varepsilon(p_0)\delta(P^2)\ \left(-m^2/\varepsilon p^2\right)^3 \tag{3.13}$$

where, for short, we do not write the remaining integration over p (see the remark after (3.20)). The imaginary term $\{-B^3\}$ is easily integrated over P^2 and yields

$$\varepsilon(p_0)\ (1+2\varepsilon)^3\ (\frac{\sin(\pi\varepsilon)}{\pi\varepsilon})^3\ (-\frac{\pi m^2}{p^2})^3\ (\frac{1}{3\varepsilon}) \tag{3.14}$$

The other term $\{3A^2B\}$ is a bit more involved. Integrated over P^2 it results in

$$\varepsilon(p_0)\ 3(\frac{\pi m^2}{p^2})(-\frac{m^2}{\varepsilon p^2})^2\ (1+2\varepsilon)\ (\frac{\sin(\pi\varepsilon)}{\pi\varepsilon})\left\{\frac{1}{\varepsilon} - \frac{2(1+2\varepsilon)\cos(\pi\varepsilon)}{2\varepsilon} + \frac{(1+2\varepsilon)^2\cos^2(\pi\varepsilon)}{3\varepsilon}\right\} \tag{3.15}$$

Expanding in powers of ε the two trigonometric functions, we get

$$\varepsilon(p_0)3(\frac{\pi m^2}{p^2})(-\frac{m^2}{\varepsilon p^2})^2\ \frac{1}{3\varepsilon}\left(1+\frac{(\pi\varepsilon)^2}{3}+O(\varepsilon^3)\right) \tag{3.16}$$

The mass singularity compensation is now made obvious as the first term in the last parenthesis of (3.16), the $+1$, compensates for the real term (3.13), whereas the second term and (3.14) cancel each other, just leaving regular contributions. Note that with respect to the previous case, $N=2$, expanding the trigonometric functions has become mandatory in order to manifest the singularity compensation.

As we have been able to check that a similar compensation of all leading and subleading mass singularities, occur at $N=4$ and $N=5$ as well, these results strongly suggest that, for all N, such an overall cancellation of mass singularities is extremely likely to take place, and not just at $N\leq 2$, as was incorrectly stated in [8]. However, the control of the ensuing finite parts and their subsequent summation over N, remains somewhat puzzling. This is why we will carry out the summation over N a different way.

First, we take the complicated expressions (2.3) and (2.5) to the much simpler forms

$$A(P^2,p^2) = -m^2\ \frac{1}{\varepsilon}\ \frac{P^2}{p^2}\{1-|\frac{P^2}{p^2}|^{\ \varepsilon}\}\ ,\qquad B(P^2,p^2) = -\pi m^2(-\frac{P^2}{p^2})^{1+\varepsilon}\Theta(-P^2) \tag{3.17}$$

Of course, with (3.17) as a crude approximation of (2.3) and (2.5), only the leading mass singularities cancel out for all N, while leaving a finite series of subleading mass singularities. The latter, though, can be resummed into regular functions. Then, bringing the expressions (3.17) to completion, that is to the forms (2.3)-(2.5), the full result can be derived rather easily out of those regular functions. This procedure will further open two interesting possibilities, dealt with in sections 5 and 6. Also, it applies straightforwardly to the (fermionic) cases where HTL self energies are proportional to the second kind Legendre function $Q_0(p_0,p)$ rather than Q_1.

We begin with showing that for all N, the stronger singularities, of order ε^{-N} cancel each other. In effect, at general N, such a singularity comes from the real part of Σ_R^N. It is

$$\left((-1)^N/N!\right)\pi\varepsilon(p_0)\ \delta^{(N)}(P^2)\ A^N(P^2,p^2) = \pi\varepsilon(p_0)\delta(P^2)\ \left(-m^2/\varepsilon p^2\right)^N \tag{3.18}$$

whereas that very contribution from the imaginary part of Σ_R^N can be written

$$-\varepsilon(p_0)\pi m^2(-\frac{m^2}{\varepsilon p^2})^{N-1}C_N^1\int_{-p^2}^{0}\frac{\mathrm{d}P^2}{(P^2)^{N+1}}\left(-\frac{P^2}{p^2}\right)^{1+\varepsilon}(P^2)^{N-1}(1-|\frac{P^2}{p^2}|^{\varepsilon})^{N-1} \tag{3.19}$$

where the C_N^k are the binomial coefficients, and where the $\Theta(-P^2)$ distribution has been turned into the $[-p^2, 0]$-integration range, by taking the condition $P^2 << p^2 << T^2$ into account. The integral over P^2 splits into N integrals, the sum of which reads [11]

$$(p^{-2})\left(\frac{1}{\varepsilon}\right)\ \left(C_N^1\ \sum_{k=0}^{N-1}\ \frac{C_{N-1}^k(-1)^k}{k+1}\ =\ 1\right) \tag{3.20}$$

By putting this expression back into (3.19), one gets the exact compensation of the ε^{-N} singularity of equation (3.18), thus completing the proof that the two most singular terms cancel each other at any number N of one-loop self energy insertions along the P-line.

It is interesting to note that, for this process at least, the choice of the two independent variables P^2 and p^2 is a most convenient one. In particular, singularity compensation is rendered manifest after a unique one dimensional integration, the one over P^2, is performed. Also, it displays the mass (or collinear) singularity character of the encountered poles. Had we choosen p_0 and $|\vec{p}|$ as another, more customary choice of two independent variables, none of these two properties were obtained.

However a milder singularity (like (3.14)) remains non cancelled. The phenomenon of course starts out at $N = 3$ with the $\{B^3\}$ term, and is, thereof, general. For any $N = 2k + 1$ with $k \geq 1$, the imaginary part of Σ_R^N develops a similar singularity under integration over P^2

$$(i\varepsilon(p_0)B)^{2k+1} \Rightarrow i\varepsilon(p_0)(-1)^k\frac{\Theta(-P^2)\mathbf{P}}{(-P^2)^{1-(2k+1)\varepsilon}}(\pi\frac{m^2}{p^{2+2\varepsilon}})^{2k+1} \Rightarrow \frac{i\varepsilon(p_0)(-1)^k}{(2k+1)\varepsilon}(\frac{\pi m^2}{p^{2+2\varepsilon}})^{2k+1} \tag{3.21}$$

In the imaginary part of Σ_R^N, but for N-values greater than $2k + 1$ at some given value of $k \geq 1$, each of these terms do appear again, multiplied by some powers of $A(P^2, p^2)$. Typically,

$$\varepsilon(p_0)(-1)^k C_N^{2k+1} A^{N-2k-1} B^{2k+1}, \qquad N \geq 2k+1, \qquad k \geq 1 \tag{3.22}$$

Selecting large enough a value of N, the sum of these terms from $\mathcal{N} = 2k + 1$ to $\mathcal{N} = N$, say, reads

$$\varepsilon(p_0)(-1)^k(-\pi\frac{m^2}{p^{2+2\varepsilon}})^{2k+1}\sum\nolimits_{\mathcal{N}}(-\frac{m^2}{\varepsilon p^2})^{\mathcal{N}-2k-1}C_{\mathcal{N}}^{2k+1}\int_{-p^2}^{0}\mathrm{d}P^2\mathbf{P}\frac{(1-|P^2/p^2|^{\varepsilon})^{\mathcal{N}-2k-1}}{(-P^2)^{1-(2k+1)\varepsilon}} \tag{3.23}$$

Carrying out the integration over P^2 and taking advantage of the arithmetical identity (see Appendix)

$$C_{\mathcal{N}}^{2k+1}(\sum_{j=0}^{\mathcal{N}-2k-1}\ \frac{(-1)^j}{j+2k+1}C_{\mathcal{N}-2k-1}^j) = \frac{1}{2k+1} \tag{3.24}$$

the expression (3.23) can be given the much simpler form

$$\varepsilon(p_0)(-1)^k(-\pi\frac{m^2}{p^{2+2\varepsilon}})^{2k+1}\ \frac{1}{(2k+1)\ \varepsilon}\ \sum\nolimits_{\mathcal{N}=2k+1}^{\mathcal{N}=N}\ (-\frac{m^2}{\varepsilon p^2})^{\mathcal{N}-2k-1} \tag{3.25}$$

Summing over N the imaginary part of $\Pi_R^{(N)}$, and letting N tend to infinity, one can re-arrange the sum of the series by singling out the $\{B^{2k+1}\}$-terms we have just analysed. One gets

$$\varepsilon(p_0)(-1)^k\ (-\pi\frac{m^2}{p^{2+2\varepsilon}})^{2k+1}\frac{1}{(2k+1)\ \varepsilon}\ \times\ Z(\varepsilon,p)\ ,\quad Z(\varepsilon,p) = \left(1-(-\frac{m^2}{\varepsilon p^2})\right)^{-1} \tag{3.26}$$

The occurence of the function Z is quite remarkable as it can be obtained out of the real part of the self energy by differentiating it with respect to P^2, at $P^2 = 0$. We have, the prime indicating such a derivation

$$Z(\varepsilon,p) = \big(1-\mathrm{Re}\Sigma'(0,p^2)\big)^{-1} = \big(1+m^2/\varepsilon p^2\big)^{-1} \tag{3.27}$$

Eventually, summing over k, we get

$$\sum_{N}^{\infty} \int \mathrm{d}P^2 \frac{\mathbf{P}}{(P^2)^{N+1}} \sum_{k=1}^{2k=N-1-\Theta((-1)^N)} C_N^{2k+1} (-1)^k B^{2k+1} A^{N-2k-1} = \varepsilon(p_0)\ \frac{Z(\varepsilon,p)}{\varepsilon} F(p) \tag{3.28}$$

where one has

$$\lim_{\varepsilon=0}\ Z(\varepsilon,p)/\varepsilon = p^2/m^2, \qquad F(p) = -\frac{1}{12}\left(\frac{g}{4}(\frac{T}{p})^{1+\varepsilon}\right)^2 + \mathrm{tg}^{-1}\left\{\frac{1}{12}\left(\frac{g}{4}(\frac{T}{p})^{1+\varepsilon}\right)^2\right\} \tag{3.29}$$

two regular functions of p in the domain $\mathcal{D}$. Note that for the series (3.28) to converge and yield the function $F(p)$, it is necessary that the condition

$$1 \geq \frac{1}{12}(\frac{gT}{4p})^2 \implies p \geq (\frac{1}{8\sqrt{3}})gT \tag{3.30}$$

be fulfilled. At its turn, (3.30) is obviously consistent with the assumed softness of the P-line. The convergence of the series displayed in (3.25) and leading to (3.27) is less obvious. This is because $m^2/\varepsilon p^2$ cannot be expected to lie within the convergence radius of (3.25) when ε^{-1} regularizes a potentially divergent behaviour. As should be made clear through sections 5 and 6, though, (3.27) can be provided with the same rigorous proof as the one developed at $T=0$, relying on the renormalization group equations and the asymptotic freedom property of the model [13].

To summarize, we have obtained

$$\mathrm{Im}\Pi_R(Q) = \frac{e^2}{(2\pi)^{D-1}}(\frac{q}{2})^{D-3}\frac{\Omega_D}{4q}(1+2n^B(\frac{q}{2}))\,(Z(\varepsilon,q/2)/\varepsilon)\,F(q/2) \tag{3.31}$$

an infrared safe result. Up to the Born term which corresponds to zero self energy insertion, $N=0$, the result (3.31) should coincide, by construction, to a calculation of $\mathrm{Im}\Pi_R(Q)$ in which a bare internal P-line has been replaced by the corresponding effective propagator ${}^*\Delta_R(P)$. When q is soft, though, this only replacement is not able to yield the full order g^2T^2 correction to the zeroth order calculation, and a similar replacement must be envisaged for the second internal line as well.

Within obvious notations, the imaginary part of the Higgs "polarisation tensor" obtained by inserting N and N' self energy corrections along the P and P' lines, see Fig.1, can be given the compact form

$$\mathrm{Im}\Pi_R^{(N,N')}(Q) = \frac{2e^2}{(2\pi)^D}\int \mathrm{d}^D P(1+2n(p_0))\{\mathrm{Im}(\Delta_R^{(N')}(P'))\mathrm{Disc}_{\mathrm{P}}(\Delta_R^{(N)}(P)) + (N \leftrightarrow N')\} \tag{3.32}$$

where $\mathrm{Im}\Delta_R^{(N')}(P')$ can be read from equation (3.3) right hand side by replacing P and N by P' and N'. Writing the contribution to $\mathrm{Im}\Pi_R^{(N,N')}(Q)$ of a generic term of $\mathrm{Im}\Sigma_R^{(N')}(P')$ (see (3.3)), one gets a rather cumbersome expression which is given in the appendix. The key observation, though, is as follows. Both P and P' lines develop singular behaviours on the light cone, but for the P' line, the condition $P'^2=0$ gets translated into the condition $\{2p-q=0\}$, and the corresponding light cone singular behaviours into generic factors

$$\Theta(2p-q)[q(2p-q)]^{-1+(2k'+1)\varepsilon} \tag{3.33}$$

The point is that integrations over P^2 and then over p, just decouple and get the two mass singular behaviours factorised. This amazing simplification is entirely due to the peculiarity of the process under consideration, with $\vec{q}=0$, and does not seem to extend beyond. Having integrated over P^2, and summed over N, one gets effectively

$$\sum_{N}^{\infty} \mathrm{Im}\Pi_R^{(N,N')}(Q) = \int \frac{\mathrm{d}^{D-1}p}{2p}(1+2n^B(p))\mathrm{Im}(\Delta_R^{(N')}(q-p,\vec{p}))\ (Z(\varepsilon,p)/\varepsilon)F(p) \tag{3.34}$$

Now the functions Z/ε and F of (3.29) are perfectly regular at $p = q/2$, so that the same pattern as before, at $N' = 0$, applies here again for $\Delta_R^{(N')}(P')$. Using the P'-adapted form of identity (3.20), that is, explicitly

$$C^1_{N'} \int_{q/2}^{q} \frac{\mathrm{d}p}{[q(2p-q)]^{1-\varepsilon}} \left(1 - |\frac{q(2p-q)}{4p^2}|^{\varepsilon}\right)^{N'-1} = \frac{q^{2\varepsilon}}{2q}\frac{1}{\varepsilon} \tag{3.35}$$

it is shown in the appendix that the two most singular terms, of order $\varepsilon^{-N'}$ compensate each other, while leaving a finite series of sub-leading mass singularities. Summing the latter series, the final result reads

$$\mathrm{Im}\Pi_R(q,\vec{0}) = \frac{e^2}{(2\pi)^{D-1}}(\frac{q}{2})^{D-3}\frac{\Omega_D}{4q}(1+2n^B(\frac{q}{2}))(Z(\varepsilon,q/2)/\varepsilon)^2 F^2(q/2) \tag{3.36}$$

It is infrared stable and in line with (3.31) on view of the peculiar $\{P, P'\}$ internal line symmetry. By construction, summing over N and N' from zero to infinity, just replaces both internal lines by their corresponding effective propagators, ${}^*\Delta_R(P)$ and ${}^*\Delta_R(P')$, so that the above result (3.36) has a simple relation with the full estimation of the Higgs polarisation tensor $\Pi_R(Q)$ imaginary part, by means of the effective perturbation theory. Explicitly,

$$\mathrm{Im}^*\Pi_R(q,\vec{0}) \simeq \ \mathrm{Im}\Pi_R^{\mathrm{Born}}(q,\vec{0}) \ + \ 2\times(3.31) \ + \ (3.36) \tag{3.37}$$

where, within standard notations, the quantity $\mathrm{Im}^*\Pi$ corresponds to the picturial representation of Fig.2, and where $\mathrm{Im}\Pi^{\mathrm{Born}}$ is just read from (3.31) or (3.36) by dropping the Z/ε and F functions. Once completed with the results of the next section, the expressions (3.31) and (3.37) will provide the basis of a future numerical analysis [18].

4- The full HTL treatment

Our previous treatment of self energy insertions is based on (3.17), that is on the leading, singular most terms in an ε expansion of the functions A and B. However, nothing guarantees that the less singular parts could be safely ignored. This possibility even appears somewhat paradoxical. Singular pieces, generated by this order ε^{-1}-part, usually cancel out in the calculation of transition rates, as we have just seen, and are accordingly not expected to be the most relevant ones. Indeed, we are carrying out the calculation a different way, and we must now show how our previous results get modified when proper account is taken of the subleading ε-expansion terms. For the sake of simplicity this is illustrated by considering zero self energy insertion along the P' line.

As a shorthand notation, we introduce v, the variable

$$v(\varepsilon) = \frac{1+2^{1+\varepsilon}\varepsilon}{4^{\varepsilon}} \ , \quad \text{or} \quad v = 1+2\varepsilon \quad \longrightarrow \quad v = 1+\kappa\varepsilon + O(\varepsilon^2) \tag{4.1}$$

The second possibility for v corresponds to the case where factors of $2^{\pm\varepsilon}$ are dropped, as we have seen in the previous section 3, whereas the last expression summarizes any of the two possibilities. The real part of the full HTL self energy reads accordingly

$$\mathcal{A}(P^2,p^2) = (-\frac{m^2}{\varepsilon p^2}P^2)\left(1 - v|\frac{P^2}{p^2}|^{\varepsilon}\left(\Theta(P^2) + \cos(\pi\varepsilon)\Theta(-P^2)\right)\right) \tag{4.2}$$

whereas the imaginary part, the function $B(P^2,p^2)$ of (2.5), can be written

$$\mathcal{B}(P^2,p^2) = -u(\varepsilon)(-P^2)^{1+\varepsilon}\Theta(-P^2) \ , \qquad u(\varepsilon) = \frac{\pi m^2}{p^{2+2\varepsilon}}\ v(\varepsilon)\ \frac{\sin(\pi\varepsilon)}{\pi\varepsilon} \tag{4.3}$$

Acting upon $\mathcal{A}^N$, the terms involving $\delta^{(N)}$ distributions are left the same as before, whereas the only changes come from the $\mathrm{Im}\{\Sigma^N\}$-pieces which now read

$$\mathrm{Im}\{\Sigma^N\} = \varepsilon(p_0) \sum_{k=0}^{2k=N-1-\Theta((-1)^N)} (-1)^k C_N^{2k+1} \mathcal{A}^{N-2k-1}\mathcal{B}^{2k+1} \tag{4.4}$$

and where the sum over k is extended to $k=0$ so as to cover the new singular and regular perturbative contributions induced by the new functions $\mathcal{A}$ and $\mathcal{B}$. Summing over N, and taking the large N limit, one has to control the extra pieces which are generated when ε is eventually sent to zero. Once folded with the corresponding principal value distribution of (3.3), the building blocks of (4.4) yield

$$\sum\nolimits_{N,k} (-1)^k \left(u(\varepsilon)\right)^{2k+1} \ \frac{1}{\varepsilon} \ \left(C_N^{2k+1} \sum_{j=0}^{N-2k-1} \frac{(-1)^j v^j(\varepsilon) \cos^j(\pi\varepsilon)}{j+2k+1} C^j_{N-2k-1} \right) \ (-\frac{m^2}{\varepsilon p^2})^{N-2k-1} \tag{4.5}$$

where a sign $\varepsilon(p_0)$ is understood, as well as the range $\{1 \leq N \leq \infty\}$. Let $\widehat{\Pi}(\cos(\pi\varepsilon))$ be that very expression between parenthesis in (4.5), and first calculate $\widehat{\Pi}(\cos(\pi\varepsilon))$ at $\cos(\pi\varepsilon) = 1$, that is, $\widehat{\Pi}(1)$. With respect to the previous calculation, equations (3.21) to (3.25), the whole change is thus translated into the occurence of the new factors $u(\varepsilon)$, as defined in (4.3), and $\{v^j \cos^j(\pi\varepsilon)\}$ between the parenthesis. Setting $\cos(\pi\varepsilon) = 1$, the corrections to (3.21)-(3.25), can be obtained by Taylor expanding $\widehat{\Pi}(1)$ in powers of ε, writing

$$\widehat{\Pi}(1) = \sum_{n=0}^{\infty} \frac{(\kappa\varepsilon)^n}{n!} \widehat{\Pi}^{(n)}(1), \qquad \widehat{\Pi}^{(0)}(1) = \frac{1}{2k+1} \tag{4.6}$$

where the value of $\widehat{\Pi}^{(0)}(1)$ is given by (3.24). Unfortunately, when doing so, the coefficients $\widehat{\Pi}^{(n)}$ result in finite series of Stirling numbers of the first kind. While instructive for some respects (next section 6 and appendix), the resummation of the ensuing series is practically undoable [23], but for a few (three) Stirling numbers. This suggests to follow a different approach. Indeed, a first order inhomogeneous differential equation in the variable v is readily found for $\widehat{\Pi}(1)$, that we hereafter define as $\Pi(v)$. Taking the appropriate "boundary condition" into account, the solution reads

$$\Pi(v) = \frac{1}{v^{2k+1}} \ \left(\Pi(1) + C_N^{2k+1} \int_1^v \mathrm{d}x \ x^{2k}(1-x)^{N-2k-1} \right) \ , \qquad \Pi(1) = \frac{1}{2k+1} \tag{4.7}$$

where the value of $\Pi(1)$, the "boundary condition" is, again, fixed by (3.24). It is clear that the first term of (4.7), proportional to $\Pi(1)$, just reproduces a result identical to (3.31), that is

$$\left(\frac{Z}{\varepsilon} = \frac{p^2}{m^2}\right) \ \mathrm{tg}^{-1}\left(\frac{u(\varepsilon)}{v(\varepsilon)}\right) \tag{4.8}$$

whereas the second term of (4.7) yields

$$\sum\nolimits_{N,k} (-1)^k \left(\frac{u(\varepsilon)}{v(\varepsilon)}\right)^{2k+1} \ \frac{1}{\varepsilon} \ \left(C_N^{2k+1} \int_1^v \mathrm{d}x \ x^{2k}(x-1)^{N-2k-1} \right) \ (+\frac{m^2}{\varepsilon p^2})^{N-2k-1} \tag{4.9}$$

Then, relying on an average value theorem for the integral over x, we know that there exists some number $c(\varepsilon)$ such that (4.9) reads

$$\frac{1}{c(\varepsilon)} \sum\nolimits_{N,k} (-1)^k \left(\frac{u(\varepsilon)c(\varepsilon)}{v(\varepsilon)}\right)^{2k+1} (\frac{p^2}{m^2}) \ C_N^{2k+1} \ \frac{\left[m^2\left(v(\varepsilon)-1\right)/\varepsilon p^2\right]^{N-2k}}{N-2k} \ , \qquad 1 \leq c(\varepsilon) \leq v(\varepsilon) \tag{4.10}$$

To proceed further, it is possible to take advantage of the trick introduced in the appendix, equation (A.1), and one finds

$$\frac{1}{c(\varepsilon)} \ \frac{p^2}{m^2} \left\{ \mathrm{tg}^{-1}\left(\frac{u(\varepsilon)c(\varepsilon)}{v(\varepsilon)} \left(1 - \frac{m^2(v-1)}{\varepsilon p^2}\right)^{-1} \right) - \mathrm{tg}^{-1}\left(\frac{u(\varepsilon)c(\varepsilon)}{v(\varepsilon)}\right) \right\} \tag{4.11}$$

In order to get (4.11) out of (4.10), series have been summed up, which are convergent within a radius specified by the condition

$$p^4/m^4 \geq \pi^2 + 2^2 \tag{4.12}$$

The limit $\varepsilon = 0$ can be taken safely. One has $v(0) = 1$, and likewise, $c(0) = 1$ by virtue of (4.10). Adding up (4.8) and (4.11), the full result reads

$$\left(\frac{p^2}{m^2}\right) \ \mathrm{tg}^{-1}\left(\frac{\pi m^2}{p^2 - \kappa m^2}\right) \tag{4.13}$$

In the calculation of $\mathrm{Im}\Pi_R(Q)$, p is fixed at $q/2$ by the kinematics and so, including the Born term as a constant that we do not write for short, one gets eventually

$$\{C^{st}\}^{-1}\mathrm{Im}\Pi_R(Q) = (\frac{q^2}{4m^2}) \ \mathrm{tg}^{-1}\left(\frac{4\pi m^2}{q^2 - 4\kappa m^2}\right) \tag{4.14}$$

Now, one must envisage the change in the calculations of (4.5) based on $\widehat{\Pi}(\cos(\pi\varepsilon))$ instead of $\widehat{\Pi}(1)$. Indeed, one can show that (4.8), (4.11) and (4.14) are left the same, but for terms of order ε at most. In the calculation of (4.5) based on $\widehat{\Pi}\,(\cos(\pi\varepsilon))$ instead of $\widehat{\Pi}(1)$, we get, as a first non trivial correction to $\Pi(1)$,

$$\{\frac{-(\pi\varepsilon)^2}{2!\ \varepsilon}\}\sum\nolimits_{N,k} (-1)^k\,(u(\varepsilon))^{2k+1}\left(C_N^{2k+1}\sum_{j=0}^{N-2k-1}\frac{(-1)^j\ jv^j}{j+2k+1}C^j_{N-2k-1}\right)\ (-\frac{m^2}{\varepsilon p^2})^{N-2k-1} \tag{4.15}$$

that is,

$$\{\frac{-(\pi\varepsilon)^2}{2!\ \varepsilon}\}\sum_{N,k}(-1)^k u(\varepsilon)^{2k+1}C_N^{2k+1}(\frac{\kappa m^2}{p^2})^{N-2k-1} - \{\frac{-(\pi\varepsilon)^2}{2!}\}\frac{p^2}{m^2}[u(\varepsilon)\frac{\partial}{\partial u(\varepsilon)}]\ \Big\{(4.8)+(4.11)\Big\} \tag{4.16}$$

The second term of order ε^2 is obviously finite. The first one can be calculated and is finite too. One gets

$$\{-\frac{\pi^2}{2!}\varepsilon\}\ \left(\frac{\pi m^2 p^2}{(p^2-\kappa m^2)^2+\pi^2 m^4} - \frac{\pi m^2}{p^2}\right)\ +\ O(\varepsilon^2) \tag{4.17}$$

Then, discriminating between the cases j even or odd (for the expansion of $\cos^j(\pi\varepsilon)$), a similar conclusion is reached for the subsequent, higher order ε-corrections to $\widehat{\Pi}(1)$. Alternatively, one may proceed a more global way, and try to express $\widehat{\Pi}\,(\cos(\pi\varepsilon))$ as a function of $\widehat{\Pi}(1)$ and $\cos(\pi\varepsilon)$. A first order inhomogeneous differential equation satisfied by $\widehat{\Pi}\,(\cos(\pi\varepsilon))$ can be derived, the solution of which reads (with the appropriate "boundary condition", that is $\widehat{\Pi}(1) = \Pi(v)$),

$$\widehat{\Pi}(\cos(\pi\varepsilon)) = \frac{1}{\cos^{2k+1}(\pi\varepsilon)}\ \left(\Pi(v) + \frac{C_N^{2k+1}}{v^{2k+1}}\int_v^{v\cos(\pi\varepsilon)} \mathrm{d}x\ x^{2k}(1-x)^{N-2k-1}\right) \tag{4.18}$$

Using (4.18), it is straightforward to check that the same result as (4.17) is obtained.

5- The mixing of singularities

A well known feature of real time formalisms is the fact that $T = 0$ and $T \neq 0$ parts come out additive. This property of first non trivial order of ordinary perturbation theory does not persist at higher orders which mix up zero and non-zero temperature contributions. When the latter are singular, the situation is the one of "singularity mixing", an admittedly complicated one [9,12], which, at least at the author knowledge, has

not been much investigated until now. In the present situation though, the singularity mixing problem turns out to be unexpectedly simplified and this is worth emphasizing.

At $T = 0$, the one loop order self energy is well known [9,13]

$$\Sigma(P^2, \mu^2) = \frac{\alpha}{2} B(\frac{D}{2} - 1, \frac{D}{2} - 1)\ P^2 (-\frac{P^2}{\mu^2})^{\frac{D}{2}-3}\ , \qquad \alpha = \frac{g^2}{(4\pi)^{D/2}} \tag{5.1}$$

Renormalized at mass scale μ, one therefore has, writing $\Sigma_F = a + ib$,

$$a(P^2, \mu^2) = -\frac{\alpha}{12\varepsilon} P^2 \left(1 - |\frac{P^2}{\mu^2}|^\varepsilon \left(\Theta(-P^2) + \Theta(P^2)\cos(\pi\varepsilon)\right)\right), \qquad b(P^2) = -\frac{\pi\alpha}{12} \left(P^2\right)^{1+\varepsilon} \Theta(P^2) \frac{\sin(\pi\varepsilon)}{\pi\varepsilon} \tag{5.2}$$

In the function $a(P^2, \mu^2)$, the first term, given by the $+1$ in the parenthesis, is the ultraviolet counter-term of the $T = 0$ renormalization procedure. When differentiated with respect to P^2, though, the same factor $1/\varepsilon$ is to be understood as the dimensional regularization of an infrared (collinear) singularity, as is made obvious by simple power counting arguments.

Now, a $T \neq 0$ real time formalism can be used to calculate pure $T = 0$ quantities, just by letting T tend to zero. This long known property of real time formalisms [14] has been explicitly verified in [9] at second non trivial order of perturbation theory. Thus, in principle, the global one loop self energy correction can be inserted in our previous equations, so as to get by the same token, both zero and non zero temperature corrections. In practice however, the different analytical properties involved at $T = 0$ and $T \neq 0$, have long made this property short of any practical purpose.

Here instead, we will take advantage of the results of previous sections 3 and 4, dealing first with the functions A and B of section 3. Then, remarking that D_u, the infinitesimal dilatation operator in the variable $u_{\varepsilon=0}(p)$, ignores both $Z(\varepsilon, p)/\varepsilon$ and $T = 0$ variables, the results are made complete by acting upon them with D_u,

$$\exp\left(-\ln\left(1 - \frac{m^2(v(\varepsilon) - 1)}{\varepsilon p^2}\right)\ D_u\right)\ \mathrm{tg}^{-1}(u) = \mathrm{tg}^{-1}\left(u\left(1 - \frac{\kappa m^2}{p^2}\right)^{-1}\right)\ , \quad D_u = u\frac{\partial}{\partial u} \tag{5.3}$$

Thus, one can re-write those relevant parts of (2.3) and (2.5) the following way

$$A(P^2, p^2) = T^{2\varepsilon}\ (\frac{2\pi T}{p})^2 \left(-\frac{\alpha}{12\varepsilon} P^2 \left(1 - |\frac{P^2}{p^2}|^\varepsilon (\Theta(P^2) + \Theta(-P^2)\cos(\pi\varepsilon))\right)\right) \tag{5.4}$$

and

$$B(P^2, p^2) = (\frac{T}{p})^{2\varepsilon}\ (\frac{2\pi T}{p})^2 \left(-\frac{\pi\alpha}{12}(-P^2)^{1+\varepsilon}\Theta(-P^2)\frac{\sin(\pi\varepsilon)}{\pi\varepsilon}\right) \tag{5.5}$$

A global one loop renormalized (Feynman) self energy can accordingly be defined as

$$\widehat{\Sigma}_F(P^2, p^2, \mu^2) = \widehat{A}(P^2, p^2, \mu^2) + i\widehat{B}(P^2, p^2) \tag{5.6}$$

with

$$\widehat{A}(P^2, p^2, \mu^2) = a(P^2, \mu^2) - (\frac{2\pi T}{p})^2\ a(-P^2, p^2) \tag{5.7}$$

$$\widehat{B}(P^2, p^2) = b(P^2) + (\frac{2\pi T}{p})^2\ b(-P^2) \tag{5.8}$$

Folded with the principal value distributions, over their respective kinematical domains, the functions $b(P^2)$ and $b(-P^2)$ involve the same pole structure, and, up to terms of order $(p^2/\mu^2)^\varepsilon$, the associated residues are the same. In these topologies however, these terms do not regularise any singular behaviour and their $\varepsilon = 0$ limit can be taken with the alluded result (likewise, for the same reason, writing (5.7) and (5.8), the $T^{2\varepsilon}$ and $(T/p)^{2\varepsilon}$ terms of (5.3) and (5.4) have been ignored).

As a consequence, the whole series we have just analysed in the pure thermal case, goes through, unchanged, in the global treatment of both $T=0$ and $T\neq 0$ parts, but for the two replacements

$$\frac{m^2}{\varepsilon p^2} \Longrightarrow \frac{\alpha}{12\varepsilon}\left(1-\left(-(\frac{2\pi T}{p})^2\right)\right)$$

$$u_{\varepsilon=0}(p) = \frac{\pi m^2}{p^2} \Longrightarrow \frac{\pi m^2}{p^2}(1-\frac{m^2(v(\varepsilon)-1)}{\varepsilon p^2})^{-1} \Longrightarrow \frac{\pi\alpha}{12}\left(1+(\frac{2\pi T}{p})^2(1-\frac{m^2(v(\varepsilon)-1)}{\varepsilon p^2})^{-1}\right) \tag{5.9}$$

Result (4.14), for example, is transformed into

$$\{C^{st}\}^{-1}\mathrm{Im}\widehat{\Pi}_R(Q) = \left(\frac{\widehat{Z}(\varepsilon,q/2)}{\varepsilon}\right)\widehat{F}(q/2) \tag{5.10}$$

where we have defined the global functions (adding up the Born term)

$$\widehat{F}(p) = \mathrm{tg}^{-1}\left\{\frac{\pi\alpha}{12}\left(1+(\frac{2\pi T}{p})^2\ (1-\frac{m^2(v(\varepsilon)-1)}{\varepsilon p^2})^{-1}\right)\right\} \tag{5.11}$$

$$\widehat{Z}^{-1}(\varepsilon,p) = 1+\frac{\alpha}{12\varepsilon}\left(1+(\frac{2\pi T}{p})^2\right) \tag{5.11}$$

Expressions (5.11) and (5.12) are obviously reminiscent of the elementary first order additivity of the $T=0$ and $T\neq 0$ parts, while displaying its full one loop leading order (HTL) realization. Of course, we do not expect that this simple pattern should extend beyond the leading HTL approximation. Clearly, the HTL peculiarity is at play (see remark (i) below). By construction, the thermal contributions of (5.11) and (5.12) are valid for soft internal momenta and are thus enhanced by a factor $1/g^2$ with respect to the $T=0$ ones. Some remarks are in order.

(i)-The striking similarity between the renormalized $T=0$ self energy and its $T\neq 0$, HTL counter part is worth emphasizing (compare equations (5.2) with (5.4) and (5.5)). It is at the origin of a possible global treatment, as we have just sketched. Unfortunately, we have been unable to find any simple interpretation of this amazing fact, except that it is certainly in line with [21].

(ii)-Things can be viewed the other way round. Once it is verified that the couples of functions (a,b) and (A,B) are involved in phase space domains over which they develop the same singular structures, then, (5.7) and (5.8) obviously indicate that the issues of singularity compensations will be the same in either cases. For the topologies considered here, where only mass singularities show up, the above statement is nothing but an alternative form of a Niegawa's recent result [15].

6. Renormalization constants

In the past twelve years, efforts have been sometimes devoted to the possibility of defining renormalization constants in a thermal context [16]. When fermionic fields are involved, for example, such a notion could be shown to be almost unreliable [16], whereas more room was left in the case of Bose-Einstein statistical fields [17]. However, even in the simpler scalar case, some ambiguity was left regarding its definition.

According to [17], the external leg renormalisation constant should be defined as

$$Z(p^2) = \mathrm{Re}\left(1-\Sigma'(0,p^2)\right)^{-1} \tag{6.1}$$

where the P^2-derivative indicated by the prime must be identified with a total derivative, kinematics rendering p^2 a function of P^2, process dependent, though weakly. At $T=0$, partial and total derivatives coincide because of Lorentz invariance, and the definition of Z reads instead

$$Z = \left(1-\mathrm{Re}\Sigma'(0)\right)^{-1} \tag{6.2}$$

Besides the familiar lack of Lorentz invariance inherent to non zero temperatures, and reflected in the p-momentum dependence, definition (6.1) differs significantly from (6.2) by the very place where the real part prescription, Re, should be taken. Furthermore, it is worth remarking that, if pertinent, (6.1) should also apply at $T = 0$ for unstable particles [17]. In a few particular cases both definitions agree. For massless fields within the dimensional regularization scheme (see (2.3) and (2.5)), for example, and also in the case of a real valued self energy function $\Sigma(P^2, p^2)$, because of the $+i\varepsilon_\alpha p_0$ advanced or retarded prescription which completes the dressed propagator determination. But the one loop self energy is not real valued in general.

Indeed, the identification (6.1) is obtained by a formal resummation of the terms involving $\delta^{(N)}(P^2)$-distributions only, as they appear on the right hand side of (3.3). Now, our analysis has shown that these contributions can not be disentangled from those involving principal value distributions, folded with imaginary parts of self energy insertions. When the latter are non zero, that is, when the self energy is complex valued, it was recognized in [17] that an expansion in the number of self energy insertions was necessary in order to give a well defined meaning to the calculations.

The analysis of the previous sections relies on such an expansion and is accordingly able to identify

$$Z(p^2) = \left(1 - \mathrm{Re}\Sigma'(0, p^2)\right)^{-1} \tag{6.3}$$

rather than (6.1), as the correct renormalisation constant, thermal counterpart (see the appendix).

Appendix

On section 3

As the arithmetical identity (3.24) is used extensively throughout the article, we find worthwhile to give its demonstration, the more as a proof by induction appears hopeless. Rather we can write

$$(3.24) = C_N^{2k+1}\Big(\sum_{j=0}^{N-2k-1} (-1)^j C^j_{N-2k-1}\Big) \int_0^1 \mathrm{d}x\ x^{j+2k} \tag{$A.1$}$$

Now, interchanging the sum with the integral, one obtains the alluded identity as

$$C_N^{2k+1} \int_0^1 \mathrm{d}x\ x^{2k}(1-x)^{N-2k-1} = C_N^{2k+1} B(2k+1, N-2k) = \frac{1}{2k+1} \tag{$A.2$}$$

where $B(x, y)$ is the Euler beta-function. A whole series of similar arithmetical identities can be obtained that way.

Here are given the basic but rather cumbersome expressions leading from (3.32) to (3.36) . From (3.32), let us write explicitly

$$\mathrm{Im}\left(\Delta_R^{(N')}(P')\right) = \mathbf{P}\frac{\mathrm{Im}\Sigma_R^{N'}(P')}{(P'^2)^{N'+1}} - \pi\frac{(-1)^{N'}}{N'!}\varepsilon(p'_0)\delta^{(N')}(P'^2)\mathrm{Re}\Sigma_R^{N'}(P') \tag{$A3$}$$

Consider $\mathrm{Im}\Pi_R^{(N,N')}(Q)$, with N' fixed, and write the contribution of a generic term appearing in $\mathrm{Im}\Sigma_R^{N'}(P')$. One gets

$$\int \frac{\mathrm{d}^{D-1}p}{2p_0(P^2,p^2)}(1+2n^B(p_0)) \sum_{k'=0}^{2k'=N'-1-\Theta((-1)^{N'})} C_{N'}^{2k'+1}(-1)^{k'}\varepsilon(p_0')(-\frac{\pi m^2}{p'^{2+2\varepsilon}})^{2k'+1}(-\frac{m^2}{\varepsilon p'^2})^{N'-2k'-1}$$
$$\int \mathrm{d}P^2 \frac{\Theta(-P'^2)\mathbf{P}}{(-P'^2)^{1-(2k'+1)\varepsilon}}(1-|\frac{P'^2}{4p'^2}|^{\varepsilon})^{N'-2k'-1}\Big\{-\pi\delta(P^2)(-\frac{m^2}{\varepsilon p^2})^N - \frac{\mathbf{P}}{(P^2)^{N+1}}$$
$$\sum_{k=0}^{2k=N-1-\Theta((-1)^N)} C_N^{2k+1}(-\frac{m^2}{\varepsilon p^2})^{N-2k-1}\left(P^2(1-|\frac{P^2}{4p^2}|^{\varepsilon})\right)^{N-2k-1}$$
$$(-1)^k(-\frac{\pi m^2}{p^{2+2\varepsilon}})^{2k+1}((-P^2)^{1+\varepsilon})^{2k+1}\Theta(-P^2)\Big\} \qquad (A4)$$

In the light cone neighbourhood, the sum over P^2 develops mass singular behaviours , whereas the remainder of the integrand is a well behaved function. The singular part of $(A4)$ is accordingly given by

$$\int \frac{\mathrm{d}^{D-1}p}{2p}(1+2n^B(p))C_{N'}^{2k'+1}(-1)^{k'}\varepsilon(q-p)(-\frac{\pi m^2}{p^{2+2\varepsilon}})^{2k'+1}(-\frac{m^2}{\varepsilon p^2})^{N'-2k'-1}$$
$$\Theta(2p-q)[q(2p-q)]^{-1+(2k'+1)\varepsilon}\left(1-|\frac{q(2p-q)}{4p^2}|^{\varepsilon}\right)^{N'-2k'-1}\int \mathrm{d}P^2\Big\{-\pi\delta(P^2)(-\frac{m^2}{\varepsilon p^2})^N$$
$$-\frac{\mathbf{P}}{(P^2)^{N+1}}C_N^{2k+1}(-\frac{m^2}{\varepsilon p^2})^{N-2k-1}\left(P^2(1-|\frac{P^2}{4p^2}|^{\varepsilon})\right)^{N-2k-1}$$
$$(-1)^k(-\frac{\pi m^2}{p^{2+2\varepsilon}})^{2k+1}((-P^2)^{1+\varepsilon})^{2k+1}\Theta(-P^2)\Big\} \qquad (A5)$$

where we have used the relation $\vec{p}=\vec{p}'$, particular to both the physical process under consideration with $Q=(q,\vec{0})$, and our notations, specified on Fig.1. Integrating now over P^2, the singular most terms of order ε^{-N} cancel out thanks to (3.20), leaving a finite series of sub-leading mass singularities. The latter can be summed over N, using (3.24), so as to yield eventually

$$\sum_N^{\infty}(A2) = \int \frac{\mathrm{d}^{D-1}p}{2p}(1+2n^B(p)) \sum_{k'=0}^{2k'=N'-1-\Theta((-1)^{N'})} C_{N'}^{2k'+1}(-1)^{k'}\varepsilon(q-p)(-\frac{\pi m^2}{p^{2+2\varepsilon}})^{2k'+1}$$
$$(-\frac{m^2}{\varepsilon p^2})^{N'-2k'-1}\frac{\Theta(2p-q)}{(q(2p-q))^{-1+(2k'+1)\varepsilon}}(1-|\frac{q(2p-q)}{4p^2}|^{\varepsilon})^{N'-2k'-1}(\frac{Z(\varepsilon,p)}{\varepsilon})F(p) \qquad (A6)$$

The integral over $\vec{p}$ factorizes the total solid angle of the model. Then, the integration over $|\vec{p}|$ ranges from $p=q/2$ to $p=\infty$ because of the $\Theta(2p-q)$ distribution, but the sign distribution $\varepsilon(q-p)$ splits the integral into two parts. The second part, say, with an integration range from $p=q$ to $p=\infty$, will be discarded because, then, one is leaving the domain of softness of the P,P' lines, that is, the realm of relevance of the effective perturbation theory. Even though, it is easy to check that this neglected integral is perfectly regular. The first part, with an integration range from $p=q/2$ to $p=q$, exhibits light cone singular behaviours at the $p=q/2$ lower bound, through the factors (3.33). Now, by using (3.35), it is straightforward to show that the singular most contribution, of order $\varepsilon^{-N'}\times(q^2/4m^2)\times F(q/2)$ coming from the term $k'=0$ in (A6), cancels out against the contribution coming from the $\mathrm{Re}\Sigma_R^{N'}(P')$ term of $(A3)$ exactly like for the P line

$$-\pi\frac{(-1)^{N'}}{N'!}\varepsilon(p_0')\delta^{(N')}(P'^2)\mathrm{Re}\Sigma_R^{(N')}(P') \Longrightarrow -\pi\varepsilon(p_0')\delta(P'^2)\left(-\frac{m^2}{\varepsilon p'^2}\right)^{N'} \qquad (A7)$$

In (A6), one is left with the sum of $\{B^{2k'+1}\}$-terms, at $k'\geq 1$, which entail the factor $\{(q(2p-q))^{-1+(2k'+1)\varepsilon}\}$, singular at the lower boundary of the p-integration, whereas the remainder of the integrand, that is, basically,

the functions $Z(\varepsilon, p)/\varepsilon$ and $F(p)$, is perfectly regular. Then it is easy to use the arithmetical identity (3.24) in order to show that the same pattern as for the P-line applies here again, with a finite series of sub-leading mass singularity outcome. Summing the latter over N', equation (3.36) results in the limit $N' = \infty$.

On section 6

In section 3, equation (3.26), we have pointed out the occurence of a global factorizing expression $Z(\varepsilon, p)$. Together with the overall factor $1/\varepsilon$, the function Z contributes the global simple function p^2/m^2 (as seen from (3.29)), which is recovered also throughout the analysis performed insection 4. In this latter section though, the function Z is not itself immediately apparent, thereby obscuring somehow its interpretation as a renormalization constant. It is indeed an artefact of the approach followed in section 4. This is realized by getting back to the ε-expansion (4.6), for which we have $(N - 2k - 1 \geq n)$

$$\widehat{\Pi}^{(n)}(1) = C_N^{2k+1} \sum_{j=0}^{N-2k-1} \frac{(-1)^j j(j-1)..(j-n+1)}{j+2k+1} C^j_{N-2k-1} \tag{A.9}$$

Now, by an iterated use of of the arithmetical identity (3.24), and also of the identity [11]

$$\sum_{j=0}^{N-2k-1} (-1)^j \; j^{\;i-1} \; C^j_{N-2k-1} = 0, \qquad 1 \quad \leq i \leq N - 2k - 1 \tag{A.10}$$

it is easy to show that any of the $\widehat{\Pi}^{(n)}(1)$ is a polynomial of degree $(n-1)$ in the variable $(2k+1)$. Explicitly

$$\widehat{\Pi}^{(n)}(1) = \sum_{m=0}^{n-1} (-1)^m S^m_{n-1} (2k+1)^m \;, \qquad n \geq 1 \tag{A.11}$$

where the coefficients $\{S^m_{n-1}\}$ are integers, namely, Stirling numbers. This can be realized by quoting the relation, due to (A.10),

$$\widehat{\Pi}^{(n)}(1) = -[(n-1) + (2k+1)] \; \widehat{\Pi}^{(n-1)}(1), \qquad n \geq 1 \tag{A.12}$$

and this yields

$$\widehat{\Pi}^{(n)}(1) = (-1)^n \left\{ \prod_{m=0}^{n-1} (m + [2k+1]) \right\} \; \left(\frac{1}{2k+1} \right) \tag{A.13}$$

where the generating relation for the Stirling numbers of the first kind is manifest [23].

The point is that at any given order ε^n in the expansion (4.6), the coefficient $\widehat{\Pi}^{(n)}(1)$ is independent of N. On view of (4.5), we conclude that the summation over N can be carried out, and that the ensuing expression $\{Z(\varepsilon, p)/\varepsilon\}$ keeps entering the result, for all n and all k, as an overall multiplicative factor.

7. Conclusion

On the basis of a scalar model of field theory, renormalizable and asymptotically free, the light cone singular behaviour of the corresponding thermal effective perturbation theory has been investigated for some given process.

Using the HTL form of the self energy near the light cone where mass singularities develop, one can conclude that, at any number of self energy insertions, a full compensation of singularities is obtained. This conclusion brings a correction to the erroneous statement which appeared in [8], and was due to an insufficient dimensional expansion of the self energy functions. In this respect, it may be recalled that after a first series of calculations performed at first and second non trivial orders of thermal perturbation theory [3,9,19], it has been observed (and then conjectured) that singularity cancellation took place within each topology separately [20]. We can remark that our analysis supports and enforces this conjecture.

Once the detailed balance compensation of mass singularities is achieved, the result for the process under consideration, can be obtained by summing over $\{N, N'\}$ the regular contributions which are left, as this summation is mandatory in order to get the full leading order correction $O(g^2T^2)$ to the Born approximation.

Throughout the present analysis, however, the summation has been carried out a somewhat different way. Although the full HTL self energy is recognized a perfectly regular function of P^2 in any light cone neighbourhood, we have stressed that it entails a potentially singular most part, dealt with in section 3, and which turns out to yield the basic result. Remarkably enough, at the level of approximation where we have been working, this basic result enjoys two interesting properties. First, it is immediately related to the cases in which the HTL self energies are given by the (second kind) Legendre function Q_0 (fermionic fields) and is therefore relevant to the study of thermal QED and QCD, the infrared problem included [22]. Second, it is simply enough related to the case under consideration, where one has to cope with a HTL self energy on the order of the Legendre function Q_1 rather than Q_0.

Furthermore, proceeding that way, a striking analogy can be put forth between the $T = 0$ renormalized self energy and its HTL counterpart. This is certainly a remarkable and intriguing aspect which, at least in our opinion, can only be read in line with the ideas developed in [21]. In particular, the issued mass singularity structures (poles and residues), turn out to be identical, and a global treatment of both zero and non zero temperature corrections is made possible, resulting in a simple resolution of the otherwise intricate singularity mixing problem. Likewise, a new, unambiguous identification of a renormalization constant in the thermal context is obtained, and turns out to be at variance with some previous attempts.

The function $F(p)$, and *a fortiori*, its generalization $\widehat{F}(p)$, is new and interesting in this context, as it could not be obtained in previous related works [12], [18], where the effective perturbation theory was used right from the onset. Remarking the similarity of F with a first order brehmsstrahlung function, one recovers a physical picture of the HTL's which is closely related to their interpretation in terms of forward Thompson scattering amplitudes [21].

Eventually, we can stress that our guiding strategy has been to express the singular structure of an effective quantum field theory at temperature T, in terms of the same bare temperature quantum field theory. Such a strategy may be thought of as arduous. On the other hand, though, the complexity of the effective perturbation theory, the mixing of topologies it envolves, renders hazardous the control of such fine tuning mechanisms as singularity compensations, if not the very meaning of the ensuing finite results. The same approach is extended to the case of gauge field theories for timelike and lightlike external momenta [22].

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